THE UNIQUENESS OF BRAIDINGS ON THE MONOIDAL CATEGORY OF NON-COMMUTATIVE DESCENT DATA

A. L. AGORE, S. CAENEPEEL, AND G. MILITARU

Abstract. Let A be an algebra over a commutative ring k. It is known that the categories of non-commutative descent data, of comodules over the Sweedler canonical coring, of right A-modules with a flat connection are isomorphic as braided monoidal categories to the center of the category of A-bimodules. We prove that the braiding on these categories is unique if there exists a k-linear unitary map $E : A \to Z(A)$. This condition is satisfied if k is a field or A is a commutative or a separable algebra.

Introduction

Braided monoidal categories play a central role in the representation theory of quantum groups, Kac-Moody algebras, quantum field theory, topological invariants to links, knots and 3-manifolds, or non-commutative differential geometry.

A natural problem is to classify all possible braidings on a given monoidal category $\mathcal{C}$. The problem is far from being trivial as we have to compute the class of all possible natural isomorphisms $c_{C,D} : C \otimes D \to D \otimes C$, for all C, $D \in \mathcal{C}$, and this depends heavily on the structure of the objects in $\mathcal{C}$. The basic example is the following: braidings on the category of representations of a bialgebra H are parameterized by R-matrices $R \in H \otimes H$. A special role in the classification of all braidings on a given monoidal category will be played by monoidal categories $\mathcal{C}$ on which we have a *unique* braided structure. There are two typical examples of such monoidal categories: the category of all sets $(\mathcal{S}et, \times, \{*\})$ and $(\mathcal{M}_k, - \otimes_k -, k)$, the category of k-modules over a commutative ring. The only braiding on this two categories is the usual flip map. In [1] we examined braidings on the category of bimodules over an algebra A. In most cases there is no braiding at all; in particular situations, for example when A is a central simple algebra over a field k, there is a unique braiding, see [1, Theorem 2.1, Cor. 2.7].

In this note, we study braidings on the monoidal category $\mathcal{M}^{A\otimes A}$ of comodules over the Sweedler canonical A-coring $A \otimes A$. This category has several alternative descriptions: it is isomorphic to the category of descent data $\underline{\mathrm{Desc}}(A/k)$, to the category $\underline{\mathrm{Conn}}(A/k)$ of right A-modules with a flat connection as defined in noncommutative geometry [3] and to the center $\mathcal{Z}({}_A\mathcal{M}_A)$ of the monoidal category of A-bimodules, all these isomorphisms can be found in [2, Theorem 2.10].

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The center $\mathcal{Z}({}_A\mathcal{M}_A)$ is braided by construction, hence it follows that $\mathcal{M}^{A\otimes A}$, $\underline{\mathrm{Conn}}(A/k)$ and $\underline{\mathrm{Desc}}(A/k)$ are also braided: the explicit description of this braiding, called the canonical braiding, is given in [2, Corollary 2.11], see also [4, Lemma 2.2]. The aim of this note is to show that this canonical braiding is unique; we will show this in Theorem 2.6, under the assumption that there exists a k-linear unitary map $E : A \to Z(A)$. This holds true if k is a field or A is a commutative or a separable algebra over k.

1. Preliminaries

Recall from [9, Def. XI.2.1] that a monoidal category is a sixtuple $(\mathcal{C}, \otimes, I, a, l, r)$, where $\mathcal{C}$ is a category, $\otimes :\ \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ is a bifunctor, I is an object in $\mathcal{C}$, and $a :\ \otimes \circ (\otimes \times id) \to \otimes \circ (id \times \otimes)$, $l :\ \otimes \circ (u \times id) \to id$, $r :\ \otimes \circ (id \times u) \to id$ are natural isomorphisms, such that certain coherence conditions are satisfied. $\mathcal{C}$ is strict if a, l and r are the identity natural transformations; McLane's coherence theorem allows us to restrict attention to strict monoidal categories. A braiding on $\mathcal{C}$ is a natural isomorphism $c :\ \otimes \to \otimes \circ \tau$ satisfying the following compatibilities:

(B1) $\quad c_{U,\,V\otimes W} = (Id_V \otimes c_{U,\,W}) \circ (c_{U,\,V} \otimes Id_W)$

(B2) $\quad c_{U\otimes V,\,W} = (c_{U,\,W} \otimes Id_V) \circ (Id_U \otimes c_{V,\,W})$

for all $U, V, W \in \mathcal{C}$, where $\tau : \mathcal{C} \times \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ is the flip functor. A braiding c is called a symmetry if $c_{U,V}^{-1} = c_{V,U}$, for all $U,\ V \in \mathcal{C}$. A (symmetric) braided category $(\mathcal{C}, c)$ is a monoidal category $\mathcal{C}$ equipped with a (symmetric) braiding c. More details on braided categories can be found in [8], [9].

Let A be a k-algebra over a commutative ring k and let $Z(A)$ be the center of A. Unadorned $\otimes$ means $\otimes_k$ and $A^{(n)}$ will be a shorter notation for the n-fold tensor product $A\otimes \cdots \otimes A$. ${}_A\mathcal{M}_A = ({}_A\mathcal{M}_A, -\otimes_A-, A)$ is the k-linear monoidal category of A-bimodules. An A-coring C is a coalgebra in ${}_A\mathcal{M}_A$. A right C-comodule is a right A-module M together with a right A-linear map $\rho :\ M \to M \otimes_A C$ satisfying the coassociativity and the counit axioms. $\mathcal{M}^C$ is the category of right C-comodules and right C-colinear maps. For further details on corings and comodules, we refer to [5]. An important example of an A-coring is Sweedler's canonical coring $C = A\otimes A$. Identifying $(A\otimes A)\otimes_A(A\otimes A) \cong A^{(3)}$, we will view the comultiplication as a map $\Delta :\ A^{(2)} \to A^{(3)}$ given by the formula $\Delta(a \otimes b) = a \otimes 1 \otimes b$; the counit $\varepsilon : A^{(2)} \to A$ is given by $\varepsilon(a \otimes b) = ab$. For a right A-module M, we can identify $M \otimes_A (A \otimes A) \cong M \otimes A$. A right $A \otimes A$-comodule is then a right A-module M together with a k-linear map $\rho :\ M \to M \otimes A$, denoted by $\rho(m) = m_{[0]} \otimes m_{[1]}$ (summation is implicitly understood), satisfying the compatibility conditions

$$m_{[0]}m_{[1]} = m; \tag{1}$$

$$\rho(m_{[0]}) \otimes m_{[1]} = m_{[0]} \otimes 1 \otimes m_{[1]}; \tag{2}$$

$$\rho(ma) = m_{[0]} \otimes m_{[1]}a \tag{3}$$

for all $m \in M$ and $a \in A$. A morphism in $\mathcal{M}^{A\otimes A}$ is a right A-module map $f : M \to N$ such that for any $m \in M$

$$f(m)_{[0]} \otimes f(m)_{[1]} = f(m_{[0]}) \otimes m_{[1]} \tag{4}$$

The category of right $A\otimes A$-comodules is denoted by $\mathcal{M}^{A\otimes A}$. There is an adjunction pair ($F := -\otimes A$, $G := (-)^{\mathrm{co}(A\otimes A)}$) between $\mathcal{M}_k$ and $\mathcal{M}^{A\otimes A}$ defined as follows:

$$F = -\otimes A : \mathcal{M}_k \to \mathcal{M}^{A\otimes A}, \quad G = (-)^{\mathrm{co}(A\otimes A)} : \mathcal{M}^{A\otimes A} \to \mathcal{M}_k$$

where for any k-module V, $F(V) = V\otimes A$ is a right $A\otimes A$-comodule with the right A-module structure given by the right multiplication on A and the coaction

$$\rho_{V\otimes A} : V\otimes A \to V\otimes A\otimes A, \quad \rho_{V\otimes A}(v\otimes a) := v\otimes 1_A\otimes a \tag{5}$$

for all $v\in V$ and $a\in A$. If $(M,\rho)\in\mathcal{M}^{A\otimes A}$, then $G(M) = M(-)^{\mathrm{co}(A\otimes A)} := \{m\in M \mid \rho(m) = m\otimes 1_A\}$. A will be viewed as a right $A\otimes A$-comodule via the regular right action given by the multiplication on A and the right coaction is given by

$$\rho_A : A\to A\otimes A, \quad \rho_A(a) := 1_A\otimes a \tag{6}$$

for all $a\in A$. Cipolla's noncommutative descent data [7] are precisely right $A\otimes A$-comodules and Cipolla's version of the Faithfully Flat Descent Theorem can be reformulated as follows: (F,G) is a pair of inverse equivalences if A is faithfully flat over k [6, Proposition 109].

2. Braidings on the category of $A\otimes A$-comodules

A right $A\otimes A$-comodule M is also a k-module, so we can consider $F(M) = M\otimes A\in\mathcal{M}^{A\otimes A}$ via

$$(m\otimes a)\,b := m\otimes ab \qquad \rho : M\otimes A\to M\otimes A\otimes A, \quad \rho(m\otimes a) := m\otimes 1_A\otimes a \tag{7}$$

for all $m\in M$, a, $b\in A$. It easily follows from (2) and (3) that the coaction $\rho : M\to M\otimes A$ is a morphism in $\mathcal{M}^{A\otimes A}$. In [2, Prop. 2.2], we observed that a right $A\otimes A$-comodule M carries a *left* A-module structure given by

$$a\cdot m = m_{[0]}\,a\,m_{[1]} \tag{8}$$

for all $a\in A$ and $m\in M$. In particular, $M\otimes A$ is a left A-module, with left A-action $a\cdot(m\otimes b) = (m\otimes 1_A)ab = m\otimes ab$. Then $\rho : M\to M\otimes A$ is left A-linear; indeed, for any $a\in A$ and $m\in M$ we have that

$$\rho(a\cdot m) = \rho(m_{[0]}am_{[1]}) = m_{[0][0]}\otimes m_{[0][1]}am_{[1]} \overset{(2)}{=} m_{[0]}\otimes am_{[1]} = a\cdot\rho(m)$$

Take $M\in\mathcal{M}^{A\otimes A}$. It follows from (1) that the right A-action $\mu_M : M\otimes A\to M$ is a right A-linear splitting map of the coaction $\rho : M\to M\otimes A$. However, μ_M is in general not left A-linear since

$$a\cdot\mu_M(m\otimes b) = a\cdot(mb) = m_{[0]}am_{[1]}b$$

while

$$\mu_M(a\cdot(m\otimes b)) = \mu_M(m\otimes ab) = m(ab).$$

This is the major drawback in our attempt to prove the uniqueness of the braiding on $\mathcal{M}^{A\otimes A}$; in the proof of Theorem 2.6, we will need a left A-linear splitting map for ρ. We give sufficient conditions for its existence in the next lemma.

Lemma 2.1. *Let A be a k-algebra, and assume that there exists a k-linear map $E : A \to Z(A)$ such that $E(1_A) = 1_A$. For any $M \in \mathcal{M}^{A\otimes A}$, the map $\mu_M^E : M \otimes A \to M$, $\mu_M^E(m \otimes a) = m_{[0]}E(m_{[1]})a$ is a left A-linear splitting map for the coaction $\rho : M \to M \otimes A$.*

Proof. We first show that μ_M^E is left A-linear. For all $a, b \in A$ and $m \in M$, we have

$$\begin{aligned} b \cdot \mu_M^E(m \otimes a) &= b \cdot m_{[0]}E(m_{[1]})a = m_{[0][0]}b\, m_{[0][1]}E(m_{[1]})a \\ &\overset{(2)}{=} m_{[0]}b\, E(m_{[1]})\, a = m_{[0]}E(m_{[1]})\, b\, a = \mu_M^E(m \otimes ba) = \mu_M^E(b \cdot (m \otimes a)). \end{aligned}$$

Finally we show that $\mu_M^E \circ \rho = \mathrm{Id}_M$:

$$\mu_M^E(m_{[0]} \otimes m_{[1]}) = m_{[0][0]}E(m_{[0][1]})m_{[1]} = m_{[0]}E(1_A)m_{[1]} = m,$$

for all $m \in M$. □

Theorem 2.2. ([2, Cor. 2.11]) *For a k-algebra A, the category $(\mathcal{M}^{A\otimes A}, - \otimes_A -, A)$ of right comodules over Sweedler's canonical coring is symmetric monoidal. For M, $N \in \mathcal{M}^{A\otimes A}$, the coaction ρ on $M \otimes_A N$ is*

$$\rho : M \otimes_A N \to M \otimes_A N \otimes A, \quad \rho(m \otimes_A n) = m_{[0]} \otimes_A n_{[0]} \otimes m_{[1]}n_{[1]} \tag{9}$$

for all $m \in M$, $n \in N$. The unit is A and the symmetry c is given by the maps

$$c_{M,N} : M \otimes_A N \to N \otimes_A M, \quad c_{M,N}(m \otimes_A n) = n_{[0]} \otimes_A m\, n_{[1]} \tag{10}$$

for any M, $N \in \mathcal{M}^{A\otimes A}$, $m \in M$, $n \in N$.

Proof. This follows from the fact that $\mathcal{M}^{A\otimes A}$ is isomorphic to the center $\mathcal{Z}({}_A\mathcal{M}_A)$ of the category of A-bimodules, which is braided monoidal, we refer to [2] for full detail. Let us show that the braiding is a symmetry: we have that

$$\begin{aligned} c_{N,M} \circ c_{M,N}(m \otimes_A n) &= c_{N,M}(n_{[0]} \otimes_A m\, n_{[1]}) \overset{(3)}{=} m_{[0]} \otimes_A n_{[0]}m_{[1]}n_{[1]} \\ &\overset{(8)}{=} m_{[0]} \otimes_A m_{[1]} \cdot n = m_{[0]}m_{[1]} \otimes_A n \\ &\overset{(1)}{=} m \otimes_A n \end{aligned}$$

for all $m \in M$ and $n \in N$. □

Proposition 2.3. *Let A be an algebra over a commutative ring k. We know that $(\mathcal{M}^{A\otimes A}, \otimes_A, A)$ is a monoidal category, with a canonical symmetry (10). The functor $F = - \otimes A : \mathcal{M}_k \to \mathcal{M}^{A\otimes A}$ is a symmetric monoidal functor.*

Proof. For M, $N \in \mathcal{M}_k$, we have natural isomorphisms

$$\varphi_0 : A \to F(k) = k \otimes A, \qquad \varphi_0(a) = 1 \otimes a$$

$$\varphi_{M,N} : F(M) \otimes_A F(N) = (M \otimes A) \otimes_A (N \otimes A) \to F(M \otimes N) = M \otimes N \otimes A$$

$$\varphi_{M,N}(m \otimes a \otimes_A n \otimes b) = m \otimes n \otimes ab$$

Straightforward computations show that F together with this family of natural isomorphisms is a monoidal functor. In order to show that F preserves the symmetry, we have to show that the diagram

$$\begin{tikzcd}
(M\otimes A)\otimes_A (N\otimes A) \arrow[r, "c_{M\otimes A,N\otimes A}"] \arrow[d, "\varphi_{M,N}"'] & (N\otimes A)\otimes_A (M\otimes A) \arrow[d, "\varphi_{N,M}"] \\
M\otimes N\otimes A \arrow[r, "\tau_{M,N}\otimes A"'] & N\otimes M\otimes A
\end{tikzcd}$$

commutes, for all $M,N\in\mathcal{M}_k$. τ is the symmetry on $\mathcal{M}_k$, and is given by the switch map. Using (10), we compute

$$\begin{aligned}
&(\varphi_{N,M}\circ c_{M\otimes A,N\otimes A})\big((m\otimes a)\otimes_A (n\otimes b)\big)\\
&\qquad = \varphi_{N,M}((n\otimes b)_{[0]}\otimes_A (m\otimes a)(n\otimes b)_{[1]})\\
&\qquad = \varphi_{N,M}((n\otimes 1_A)\otimes_A (m\otimes ab)) = n\otimes m\otimes ab;\\
&\big((\tau_{M,N}\otimes A)\circ\varphi_{M,N}\big)\big((m\otimes a)\otimes_A (n\otimes b)\big)\\
&\qquad = (\tau_{M,N}\otimes Id_A)(m\otimes n\otimes ab) = n\otimes m\otimes ab,
\end{aligned}$$

for all $a,b\in A$, $m\in M$ and $n\in N$, as needed. □

Remark 2.4. We note that the forgetful functor $F:\mathcal{M}^{A\otimes A}\to {}_A\mathcal{M}_A$ is a strict monoidal functor. In case ${}_A\mathcal{M}_A$ is a braided monoidal category (see [1, Theorem 2.1]) the functor F is not however a braided monoidal functor. To see this, we consider K to be a commutative ring such that 2 is invertible in K, and $A={}^aK^b$ the generalized quaternion algebra having $\{1,i,j,k\}$ as a K-basis, where a, b are invertible elements in K. Then we can write down the explicit formula for the (unique) braiding on ${}_A\mathcal{M}_A$ by using the R-matrix described in [1, Example 2.10]. It follows that F is not a braided functor by considering the appropriate diagram for the pair of objects $M=N:=A$ in $\mathcal{M}^{A\otimes A}$ and checking that it is not commutative in $i\otimes_A j$.

In order to prove the uniqueness of the braiding on $\mathcal{M}^{A\otimes A}$, we will need the following Lemma.

Lemma 2.5. *Let A be a k-algebra and $a\in A$. Then the natural transformation c given by*

$$c^a_{M,N}: M\otimes_A N\to N\otimes_A M,\quad c^a_{M,N}(m\otimes_A n)=n_{[0]}\otimes_A m\,n_{[1]}\,a$$

is a braiding on the monoidal category $(\mathcal{M}^{A\otimes A}, -\otimes_A -, A)$ if and only of $a=1_A$.

Proof. If c^a is a braiding then $c^a_{A,A}: A\otimes_A A\to A\otimes_A A$, $c^a_{A,A}(x\otimes_A y)=1_A\otimes_A xya$ is an isomorphism of right $A\otimes A$-comodules and, a fortiori, of right A-modules, and this implies that a has a left inverse in A. On the other hand, evaluating both sides of (B2) to $1_A\otimes 1_A\otimes_A 1_A\otimes 1_A\otimes_A 1_A\otimes 1_A$ in the situation where $U=V=W=A\otimes A$, we find that $1_A\otimes 1_A\otimes 1_A\otimes a = 1_A\otimes 1_A\otimes a\otimes a$. Multiplying the tensor factors, we obtain that $a^2=a$ and hence $a=1_A$ since a has a left inverse. □

Now we can state and prove the main result of this note.

Theorem 2.6. *Let A be a k-algebra, and assume that there exists a k-linear map $E : A \to Z(A)$ such that $E(1_A) = 1_A$. Then there is precisely one braiding on the monoidal category $(\mathcal{M}^{A\otimes A}, -\otimes_A -, A)$, namely the canonical braiding defined in (10).*

Proof. Let c be a braiding on $\mathcal{M}^{A\otimes A}$. For morphisms $f : M \to M'$ and $g : N \to N'$ in $\mathcal{M}^{A\otimes A}$, the following diagram commutes, by the naturality of c:

$$\begin{array}{ccc} M \otimes_A N & \xrightarrow{c_{M,N}} & N \otimes_A M \\ {\scriptstyle f\otimes_A g}\downarrow & & \downarrow{\scriptstyle g\otimes_A f} \\ M' \otimes_A N' & \xrightarrow{c_{M',N'}} & N' \otimes_A M' \end{array}$$

As we have seen at the end of Section 1, $A \otimes A = F(A) \in \mathcal{M}^{A\otimes A}$, and $A \otimes A$ is also a left A-module, via (8), which takes the form $a \cdot (b \otimes c) = b \otimes ac$.
The identification $A^{(3)} \cong A^{(2)} \otimes_A A^{(2)}$ transports the isomorphism $c_{A^{(2)},A^{(2)}} : A^{(2)} \otimes_A A^{(2)} \to A^{(2)} \otimes_A A^{(2)}$ to an isomorphism $\gamma : A^{(3)} \to A^{(3)}$. Then $c_{A^{(2)},A^{(2)}}$ can be computed from γ as follows:

$$c_{A^{(2)},A^{(2)}}(a \otimes b \otimes_A a' \otimes b') = c_{A^{(2)},A^{(2)}}(a \otimes 1_A \otimes_A b \cdot (a' \otimes b')) = \gamma(a \otimes a' \otimes bb'),$$

where we identified $A^{(2)} \otimes_A A^{(2)}$ and $A^{(3)}$ in the last identity. γ is completely determined by the map

$$\delta : A^{(2)} \to A^{(3)}, \quad \delta(a \otimes b) = \gamma(a \otimes b \otimes 1_A).$$

Since γ is right A-linear, we have

$$\gamma(a \otimes b \otimes c) = \gamma(a \otimes b \otimes 1_A)c = \delta(a \otimes b)c.$$

Now we adopt the temporary notation:

$$\delta(a \otimes b) = \sum \delta^1(a \otimes b) \otimes \delta^2(a \otimes b) \otimes \delta^3(a \otimes b) \in A^{(3)}.$$

Then we have that

$$\gamma(a \otimes b \otimes c) = \sum \delta^1(a \otimes b) \otimes \delta^2(a \otimes b) \otimes \delta^3(a \otimes b)\, c$$

and

$$c_{A^{(2)},A^{(2)}}(a \otimes 1_A \otimes_A a' \otimes b') = \sum \delta^1(a \otimes a') \otimes 1_A \otimes_A \delta^2(a \otimes a') \otimes \delta^3(a \otimes a')\, b', \tag{11}$$

for all $a, a', b \in A$. $c_{A^{(2)},A^{(2)}}$ is a right $A \otimes A$-colinear map, this means that the following diagram commutes:

$$\begin{array}{ccc} A^{(2)} \otimes_A A^{(2)} & \xrightarrow{c_{A^{(2)},A^{(2)}}} & A^{(2)} \otimes_A A^{(2)} \\ {\scriptstyle \rho}\downarrow & & \downarrow{\scriptstyle \rho} \\ A^{(2)} \otimes_A A^{(2)} \otimes A & \xrightarrow{c_{A^{(2)},A^{(2)}} \otimes A} & A^{(2)} \otimes_A A^{(2)} \otimes A \end{array}$$

where $\rho :\ A^{(2)} \otimes_A A^{(2)} \to A^{(2)} \otimes_A A^{(2)} \otimes A$, the right $A \otimes A$-coaction defined in (9), is given by the formula

$$\rho(a \otimes 1_A \otimes_A a' \otimes b') = a \otimes 1_A \otimes_A a' \otimes 1_A \otimes b'.$$

Evaluating the diagram at $a \otimes 1_A \otimes_A a' \otimes 1_A$, we find that δ satisfies the equation

$$\sum \delta^1(a\otimes a')\otimes\delta^2(a\otimes a')\otimes\delta^3(a\otimes a')\otimes 1_A = \delta^1(a\otimes a')\otimes\delta^2(a\otimes a')\otimes 1_A\otimes\delta^3(a\otimes a'), \tag{12}$$

for all $a, a' \in A$. In fact, it can be shown easily that the right $A \otimes A$-colinearity of $c_{A^{(2)},\,A^{(2)}}$ is equivalent to (12), but this will not be needed.
For $a \in A$, the map $f_a : A^{(2)} \to A^{(2)}$ given by $f_a(x\otimes y) := ax\otimes y$ is a morphism in $\mathcal{M}^{A\otimes A}$. The naturality of c implies that the following diagram commutes, for all $a, b \in A$:

$$\begin{tikzcd}
A^{(2)} \otimes_A A^{(2)} \arrow[r, "c_{A^{(2)},A^{(2)}}"] \arrow[d, "f_a\otimes_A f_b"'] & A^{(2)} \otimes_A A^{(2)} \arrow[d, "f_b\otimes_A f_a"] \\
A^{(2)} \otimes_A A^{(2)} \arrow[r, "c_{A^{(2)},A^{(2)}}"] & A^{(2)} \otimes_A A^{(2)}
\end{tikzcd}$$

Evaluating this diagram at $1_A \otimes 1_A \otimes_A 1_A \otimes c$, we obtain that

$$c_{A^{(2)},\,A^{(2)}}(a\otimes 1_A\otimes_A b\otimes c) = \sum bs^1 \otimes 1_A \otimes_A a\, s^2 \otimes s^3\, c \tag{13}$$

for any a, b, $c \in A$, where $\delta(1_A \otimes 1_A) = \sum s^1 \otimes s^2 \otimes s^3 \in A^{(3)}$. This implies that δ is completely determined by $\delta(1_A \otimes 1_A)$:

$$\delta(a\otimes b) = \sum bs^1 \otimes a\, s^2 \otimes s^3. \tag{14}$$

Combining (12) and (14),

$$\sum a'\, s^1 \otimes a\, s^2 \otimes s^3 \otimes 1_A = \sum a'\, s^1 \otimes a\, s^2 \otimes 1_A \otimes s^3$$

for all $a, a' \in A$. In particular, for $a = a' = 1_A$ we obtain

$$\sum s^1 \otimes s^2 \otimes s^3 \otimes 1_A = \sum s^1 \otimes s^2 \otimes 1_A \otimes s^3$$

Multiplying the second and the third tensor factor, we find that $s = \sum s^1 \otimes s^2 s^3 \otimes 1_A$. We conclude that there exists an element $R = \sum R^1 \otimes R^2 \in A \otimes A$ such that $s = R \otimes 1_A$. Then we have that

$$\delta(a\otimes b) = \sum b\, R^1 \otimes a\, R^2 \otimes 1_A \tag{15}$$

$$c_{A^{(2)},\,A^{(2)}}(a\otimes 1_A\otimes_A b\otimes c) = \sum b\, R^1 \otimes 1_A \otimes_A a\, R^2 \otimes c \tag{16}$$

for all $a, b, c \in A$. We can easily prove that $c_{A^{(2)},\,A^{(2)}}$ as defined in (16) is an isomorphism if and only if R is invertible in the algebra $A \otimes A$, but this will not be needed.
For $M \in \mathcal{M}_A$ and $m \in M$, the map $f_m : A^{(2)} \to M \otimes A$, $f_m(a \otimes b) = ma \otimes b$, is a morphism in $\mathcal{M}^{A\otimes A}$, where $M \otimes A$ is viewed as a right $A \otimes A$-comodule via (7). From the naturality of c, it follows that the following diagram commutes, for all $M, N \in \mathcal{M}_A$, $m \in M$ and $n \in N$:

$$\begin{tikzcd}
A^{(2)} \otimes_A A^{(2)} \arrow[r, "c_{A^{(2)},A^{(2)}}"] \arrow[d, "f_m\otimes_A f_n"'] & A^{(2)} \otimes_A A^{(2)} \arrow[d, "f_n\otimes_A f_m"] \\
M\otimes A \otimes_A N \otimes A \arrow[r, "c_{M\otimes A,N\otimes A}"] & M\otimes A \otimes_A N \otimes A
\end{tikzcd}$$

Evaluating this diagram at $1_A \otimes 1_A \otimes_A 1_A \otimes a$ and using (16) we obtain that

$$c_{M\otimes A,\, N\otimes A}(m \otimes 1_A \otimes_A n \otimes a) = \sum n\, R^1 \otimes 1_A \otimes_A m\, R^2 \otimes a \tag{17}$$

for all $m \in M$, $n \in N$ and $a \in A$. This means that the braiding c is completely determined in all cofree objects $M \otimes A$ of the category $\mathcal{M}^{A\otimes A}$ by the element $R \in A \otimes A$. For $M \in \mathcal{M}^{A\otimes A}$, the coaction $\rho_M : M \to M \otimes A$ is a morphism in $\mathcal{M}^{A\otimes A}$, so the following diagram commutes, again by the naturality of c:

$$\begin{array}{ccc} M \otimes_A N & \xrightarrow{c_{M,N}} & N \otimes_A M \\ {\scriptstyle \rho_M \otimes_A \rho_N}\big\downarrow & & \big\downarrow{\scriptstyle \rho_N \otimes_A \rho_M} \\ M \otimes A \otimes_A N \otimes A & \xrightarrow{c_{M\otimes A, N\otimes A}} & M \otimes A \otimes_A N \otimes A \end{array}$$

Evaluating this diagram at $m \otimes_A n$, we find that

$$\begin{aligned} (\rho_N \otimes_A \rho_M)(c_{M,N}(m \otimes_A n)) &= c_{M\otimes A, N\otimes A}(m_{[0]} \otimes m_{[1]} \otimes_A n_{[0]} \otimes n_{[1]}) \\ &= c_{M\otimes A, N\otimes A}(m_{[0]} \otimes 1_A \otimes_A m_{[1]} \cdot (n_{[0]} \otimes n_{[1]})) \\ &= c_{M\otimes A, N\otimes A}(m_{[0]} \otimes 1_A \otimes_A n_{[0]} \otimes m_{[1]} n_{[1]}) \\ &\overset{(17)}{=} \sum n_{[0]} R^1 \otimes 1_A \otimes_A m_{[0]} R^2 \otimes m_{[1]} n_{[1]}. \end{aligned} \tag{18}$$

The multiplication map μ_N is right A-linear and splits ρ_N, and the map μ_M^E from Lemma 2.1 is left A-linear and splits ρ_M. This implies that $\mu_N \otimes_A \mu_M^E$ splits $\rho_N \otimes_A \rho_M$. Applying $\mu_N \otimes_A \mu_M^E$ to (18), we obtain that

$$\begin{aligned} c_{M,N}(m \otimes_A n) &= \sum n_{[0]} R^1 \otimes_A \mu_M^E(m_{[0]} R^2 \otimes m_{[1]} n_{[1]}) \\ &= \sum n_{[0]} R^1 \otimes_A m_{[0][0]} E(m_{[0][1]} R^2) m_{[1]} n_{[1]} \\ &\overset{(2)}{=} \sum n_{[0]} R^1 \otimes_A m_{[0]} E(R^2) m_{[1]} n_{[1]} \\ &= \sum n_{[0]} R^1 \otimes_A m_{[0]} m_{[1]} E(R^2) n_{[1]} \\ &\overset{(1)}{=} \sum n_{[0]} R^1 \otimes_A m E(R^2) n_{[1]} \\ &= \sum n_{[0]} \otimes_A R^1 \cdot (m E(R^2) n_{[1]}) \\ &= \sum n_{[0]} \otimes_A m_{[0]} R^1 m_{[1]} E(R^2) n_{[1]}, \end{aligned}$$

for any $m \in M$ and $n \in N$. In the special case where $M = N = A \otimes A$, we evaluate this formula to $a \otimes 1_A \otimes_A b \otimes c$. Using (16), we obtain that

$$\sum bR^1 \otimes 1_A \otimes_A aR^2 \otimes c = \sum b \otimes 1_A \otimes_A a \otimes R^1 E(R^2) c$$

for all $a, b, c \in A$. In particular, for $a = b = c = 1_A$, we find

$$\sum R^1 \otimes R^2 \otimes 1_A = \sum 1_A \otimes 1_A \otimes R^1 E(R^2).$$

Multiplying the second and the third tensor factors, we obtain that $R = \sum 1_A \otimes R^1 E(R^2)$, so we can conclude that $R = 1_A \otimes \alpha$, for some $\alpha \in A$. Therefore, we obtain:

$$c_{M,N}(m \otimes_A n) = n_{[0]} \otimes_A m_{[0]} m_{[1]} E(\alpha) n_{[1]} = n_{[0]} \otimes_A m E(\alpha) n_{[1]} = n_{[0]} \otimes_A m n_{[1]} E(\alpha).$$

It then follows from Lemma 2.5 that c is the canonical symmetry given by (10). □

Let us finally examine the existence of a unitary k-linear map $A \to Z(A)$.

Proposition 2.7. *Let A be an algebra over a commutative ring k. There exists a unitary k-linear map $E:\ A \to Z(A)$ in each of the following situations:*

(1) *A is commutative;*
(2) *k is a field;*
(3) *A is an augmented algebra, for example a bialgebra;*
(4) *A is a separable k-algebra.*

Proof. The first three cases are obvious. Let A be a separable algebra with separability idempotent $e = e^1 \otimes e^2$ (summation understood), i.e.

$$ae^1 \otimes e^2 = e^1 \otimes e^2 a \qquad \text{and} \qquad e^1 e^2 = 1 \tag{19}$$

for all $a \in A$. The map $E : A \to A$, $E(a) = e^1 a e^2$ meets the requirements: $E(a) \in Z(A)$ follows from the centrality condition and $E(1_A) = 1_A$ follows from the normality condition in (19). □

We end our paper with the following question: does there exist a commutative ring k and a k-algebra A for which there exists a second braiding on $\mathcal{M}^{A\otimes A}$?

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Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium
E-mail address: ana.agore@vub.ac.be and ana.agore@gmail.com

Faculty of Engineering, Vrije Universiteit Brussel, Pleinlaan 2, B-1050 Brussels, Belgium
E-mail address: scaenepe@vub.ac.be

Faculty of Mathematics and Computer Science, University of Bucharest, Str. Academiei 14, RO-010014 Bucharest 1, Romania
E-mail address: gigel.militaru@fmi.unibuc.ro and gigel.militaru@gmail.com